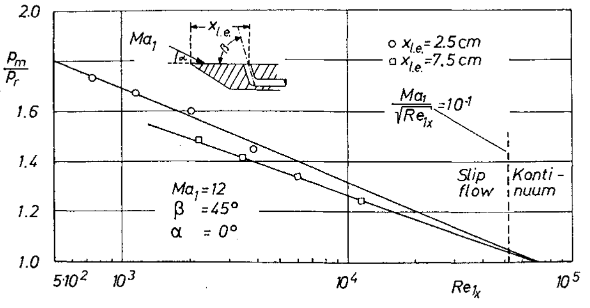

Fig. 2 p_m/p_r vs Re for inclination angle β.

mental data of two different leading edge distances $x_{l.e.}$ leads to a boundary of the measured effects. This boundary corresponds to the boundary between slip flow and continuum flow defined as $Ma/(\mathrm{Re})^{1/2} = 0.1$.

As the measured effect depends on the rarefaction of the gas, the experimental results are plotted in Fig. 3 over the rarefaction parameter $Kn_w \cdot (x_{l.e.})^{1/2}$ with $Kn_w = \lambda_w / x_{l.e.}$, $\lambda_w = 1.26(R \cdot T_w)^{1/2} \cdot \mu / p_w$ and $x_{l.e.}$ the leading edge distance. All data correlate in this plot for the different test parameters. The curve shows a sudden slope change that corresponds to the change from linear to nearly quadratic characteristic in Fig. 1. At the same point the mean free path in the cavity is of the order of the tube diameter.

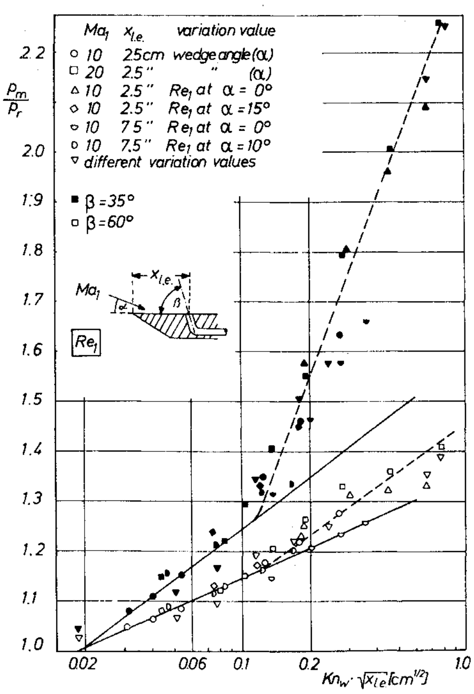

Fig. 3 Experimental results.

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Stability of a Liquid Film

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THE cooling efficiency of a liquid film developed from an ablator or a transpiration cooling system depends on its remaining in contact with the body surface, and its removal in the form of vapor only. If the interface is disturbed from its equilibrium position, waves will form and the dominant liquid layer mass loss mechanism may be the entrainment of liquid droplets into the airstream. In the absence of an external gas, an instability is provided by body forces acting outward from the liquid layer. This situation can be realized under conditions of vehicle deceleration during re-entry where the liquid experiences an effective body force directed away from the vehicle surface. The flow of an external gas has two important effects. The first is the establishment of a velocity profile via the shear at the liquid surface while the second effect is the production of perturbations in the stresses exerted by the gas on the interface due to the appearance of waves that leads to pressure perturbation[1] and shear perturbation instability mechanisms.[2] The purpose of this Note is to study the interaction of tangential body forces with pressure perturbations on the stability of liquid films. The liquid layer is assumed to be thin with one side adjacent to a solid boundary and the other side adjacent to the gas stream. A Cartesian coordinate system is introduced with the $\bar{x}$ axis coinciding with the solid interface and the $\bar{y}$ axis pointing into the liquid. The equilibrium gas-liquid interface is located at $\bar{y} = h$. The body force is directed in such a fashion as to be parallel to the direction of the flight path while the liquid motion is developed from the viscous shear stresses exerted by the gas layer on the liquid layer. The dimensionless velocity of the liquid in the positive x direction is given by

$$U = \gamma y^2 + (1 - \gamma)y \tag{1}$$

where the velocity and the vertical dimension are normalized with the interface velocity U_L, and the liquid depth h. Here

$$\gamma = g \sin\theta h^2 / 2\nu U_L,$$
$$U_L = (g \sin\theta / 2\nu)(2\tau_0 h / \rho g \sin\theta - h^2) \tag{2}$$

with g the body force per unit mass directed in such a way that $g \cos\theta$ is in the positive y-direction and g $\sin\theta$ in the

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negative x-direction, and τ_0 is the interface shear stress calculated from $\tau_0 = c_f \rho_g U_g^2$ with U_g the freestream gas velocity in the direction of the wave normals.

The primary flow just described is disturbed from equilibrium by superposing small disturbances that allow for linearization of the resulting differential equations and boundary conditions. The disturbances are represented by a stream function, $\psi(x,y,t)$, and the gas-liquid interface is assumed to have a two-dimensional traveling wave form, $\eta(x,t)$, such that

$$\psi(x,y,t) = \varphi(y)\eta(x,t), \quad \eta = \eta_0 e^{i\alpha(x-ct)} \tag{3}$$

In general c is a complex number whose real part c_r represents the dimensionless wave velocity and whose imaginary part c_i represents the damping or amplification of the disturbance depending on its sign. The parameter $\alpha = 2\pi h/\lambda$ is the dimensionless wavenumber. Substituting Eq. (3) into the equations of motion with attendent boundary conditions gives

$$\varphi^{iv} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha R[(U - c)(\varphi'' - \alpha^2\varphi) - 2\gamma\varphi] \tag{4}$$

$$\varphi(0) = \varphi'(0) = 0 \tag{5}$$

$$c = 1 + \varphi(1) \tag{6}$$

$$\varphi''(1) + \alpha^2\varphi(1) = \chi - 2\gamma \tag{7}$$

$$\varphi'''(1) - 3\alpha^2\varphi'(1) = i\Lambda + i\alpha R[T\alpha^2 - G_y - (c - 1)\varphi'(1) - (1 + \gamma)\varphi(1)] \tag{8}$$

where the following definitions have been made:

$$R = U_L h/\nu, \quad T = \sigma/\rho U_L^2 h, \quad G_y = gh\cos\theta/U_L^2$$

$$\chi\eta = (\chi_r + i\chi_i)\eta = R\tau_g/\rho U_L^2,$$

$$\Lambda\eta = (\Lambda_r + i\Lambda_i)\eta = \alpha R p_g/\rho U_L^2$$

The parameters R, T, G_y are the Reynolds, reciprocal Weber, and reciprocal Froude numbers, respectively, σ is the surface tension and τ_g and p_g are the perturbations in shear and pressure exerted by the gas on the interface which are included here following Craik.[2]

A perturbation solution for φ is sought for long wavelengths assuming that αR is small. Thus

$$\varphi = \varphi_0 + \alpha\varphi_1 + O(\alpha^2) \tag{9}$$

Equation (9) is substituted into Eq. (4), coefficients of like powers of α are collected and the results integrated subject to the boundary conditions (5), (7), and (8). The resultant expansion is then substituted into Eq. (6) to yield an equation

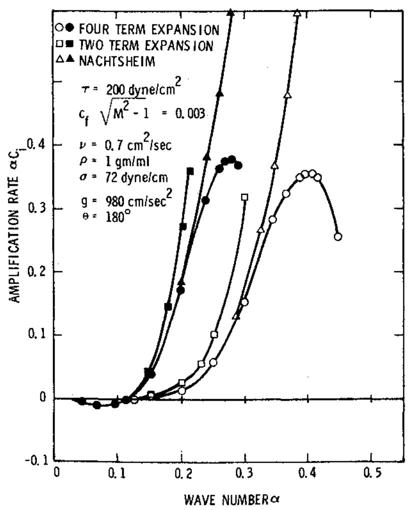

Fig. 1 Amplification rate vs wavenumber for three solution techniques. Solid symbols are for $R = 1.0$ and blank symbols are for $R = 0.1$.

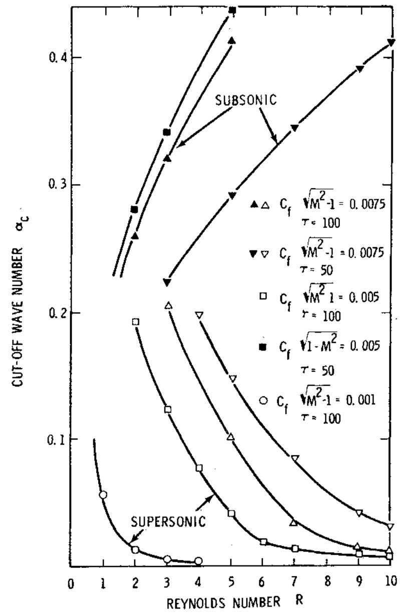

Fig. 2 Effect of liquid Reynolds number, gas Mach number and shear on cut-off wavenumbers (ρ = 1.0 g/cm³, ν = 1.0 centistoke, σ = 72 dyne/cm).

for c. The details can be found in Nayfeh and Saric[3]

$$c = 1 - \gamma + \tfrac{1}{2}\chi - \tfrac{1}{3}i\Lambda - \tfrac{1}{3}i\alpha R\{T\alpha^2 - G_y - \tfrac{9}{20}\gamma(1 - \gamma) + \tfrac{9}{40}(1 - \gamma)\chi - c[-\tfrac{5}{4}\gamma + \tfrac{5}{8}\chi - (2i/5)\Lambda] + [\gamma/21 - 13(1 - \gamma)/120]i\Lambda\} + O(\alpha^2) \tag{10}$$

Since the primary interest here is to investigate the effects of the pressure perturbations interacting with the effects of the body forces, χ will be taken to be zero in the following calculations, and Λ is calculated from the uniform inviscid flow past a wavy wall. The linearized pressure disturbance is given by

$$\Lambda_r = -\alpha^2/c_f(1 - M^2)^{1/2}, \quad \Lambda_i = 0, \quad M < 1 \tag{11}$$

$$\Lambda_r = 0, \quad \Lambda_i = \alpha^2/c_f(M^2 - 1)^{1/2}, \quad M > 1 \tag{12}$$

where the Mach number M is given by U_g/a_g since the wave speed, c_r, is very small compared to U_g.

Supersonic External Flow

In this case, Λ is given by Eq. (12) and the dimensional growth rate becomes

$$\alpha c_i \frac{U_L}{h} = \frac{\alpha^2 h^2}{90\nu^3}\left(\frac{2\tau_0}{\rho h} - g\sin\theta\right)^2 \times \left[\frac{\alpha^4}{c_f^2(M^2 - 1)} - \frac{2\alpha^2\xi_1}{c_f(M^2-1)^{1/2}} - \xi_2\right] \tag{13}$$

$$\xi_1 = \frac{15}{4}\left[Tc_r(M^2 - 1)^{1/2} - \frac{49 - 111\gamma}{168}\right],$$

$$\xi_2 = \frac{15}{2}\left(\frac{4}{5}\gamma(1 - \gamma) - G_y\right) \tag{14}$$

where c_r has been eliminated from the equation for c_i by assuming αc_i is small. The cut-off wavenumbers ($c_i = 0$) are then given by

$$\alpha_c^2 = c_f(M^2 - 1)^{1/2}[\xi_1 \pm (\xi_1^2 + \xi_2)^{1/2}] \tag{15}$$

Equation (13) shows that the wave drag is destabilizing while the surface tension is stabilizing. The component of the body force normal to the interface is stabilizing or destabilizing depending on whether it is directed from the gas to the liquid or vice versa. If the axial body force is in the same direction as the external flow ($g\sin\theta < 0$), $\gamma < 0$ from

Eq. (2), and the amplification rate increases as $|g \sin\theta|$ increases due to the increase of both of the terms inside and outside the large bracket in Eq. (13). On the other hand, if $g \sin\theta > 0$, γ is positive, and the amplification rate decreases as $g \sin\theta$ increases. The stabilizing effect of an axial body force opposing the external flow is due to the reduction of the liquid Reynolds number, and a partial cancellation of the supersonic wave drag. In order for cut-off wavenumbers to exist, ξ_1 and $\xi_1^2 + \xi_2$ must be positive. Moreover, there will be one or two cut-off wavenumbers depending on whether ξ_2 is positive or negative. The details of specific calculations are contained in Nayfeh and Saric.[3]

Let us compare our results with the numerical results of Nachtsheim.[4,5] If the axial body force is neglected and if the normal component of the body force is directed toward the liquid (i.e., $\theta = 180°$), the problem discussed in this section reduces to that studied by Nachtsheim who obtained a numerical solution of the full eigenvalue problem. In order to improve the calculations for this case, the two-term expansion for the stream function in Eq. (9) was extended to four terms. The dimensionless wave speed and amplification rate are given by

$$c_r = 1 + \frac{1}{3}\Lambda_i - \frac{2}{15}\alpha R\Lambda_i c_i - \frac{3}{5}\alpha^2\Lambda_i - \frac{1}{30}\alpha^2R^2\left\{\frac{13}{12}(T\alpha^2 - G_y) + \frac{823}{6048}\Lambda_i - \left[4(T\alpha^2 - G_y) + \frac{205}{224}\Lambda_i\right]c_r + \frac{34}{21}\Lambda_i(c_r^2 - c_i^2)\right\} + \frac{47}{90}\alpha^3R\Lambda_i c_i + \frac{1}{5040}\alpha^3R^3c_i\left\{\frac{1551}{10}(T\alpha^2 - G_y) + \frac{1021}{36}\Lambda_i - \left[3808(T\alpha^2 - G_y) + \frac{3439}{18}\Lambda_i\right]c_r + \frac{39923}{360}\Lambda_i(3c_r^2 - c_i^2)\right\} + O(\alpha^4) \quad (16a)$$

and

$$c_i = -\frac{1}{3}\alpha R\left(T\alpha^2 - G_y + \frac{13}{120}\Lambda_i - \frac{2}{5}\Lambda_i c_r\right) + \frac{1}{30}\alpha^2R^2c_i\left[4(T\alpha^2 - G_y) + \frac{205}{224}\Lambda_i - \frac{68}{21}\Lambda_i c_r\right] + \frac{1}{45}\alpha^3R\left[27(T\alpha^2 - G_y) + \frac{101}{14}\Lambda_i - \frac{47}{2}\Lambda_i c_r\right] + \frac{1}{5040}\alpha^3R^3\left\{\frac{143}{6}(T\alpha^2 - G_y) + \frac{6917}{2376}\Lambda_i - \left[\frac{1551}{10}(T\alpha^2 - G_y) + \frac{1021}{36}\Lambda_i\right]c_r + \left[1904(T\alpha^2 - G_y) + \frac{3439}{36}\Lambda_i\right](c_r^2 - c_i^2) - \frac{39923}{360}\Lambda_i(c_r^3 - 3c_rc_i^2)\right\} + O(\alpha^4) \quad (16b)$$

A direct comparison of solutions of Eqs. (16) with the two term expansion and Nachtsheim[5] is shown in Fig. 1, which is a calculation of the growth rate as a function of wavenumber at two Reynolds numbers with all other conditions held constant. The range of parameters is chosen to be within the experimental data of Marshall and Saric.[6] In this range, all these methods predict the first cut-off wavenumber very closely; however, the two term expansion breaks down shortly after that. Equation (16) continues to be accurate enough to calculate growth rates for larger α but breaks down shortly after the two term expansion. Neither small wavenumber expansion can predict the maximum growth rate or the second cut-off wavenumber which occurs at $\alpha = O(1)$. The four term expansion was used to calculate cut-off wavenumbers for the experimental conditions of Marshall and Saric.[6] It was found that the observed wavelengths were in the unstable range predicted by Eq. (16) and that the calculated wave speeds were correct to the same order. Furthermore, the trends of varying R and τ_0 were correctly predicted. It appears, therefore, that the aforementioned theory is a good approximation of the solution to the Orr-Sommerfeld equation in the region of interest and it can model the experimental results.

Figure 2 shows that in the supersonic case, an increase in R or τ_0, or a decrease in M results in a decrease in the cut-off wavenumber, and hence destabilizing. The stabilizing effect of increasing M is due to the reduction of the wave drag. The destabilizing effects of R and τ_0 can be seen by rewriting T and G in the following form:

$$T = (\sigma\nu^{-1}\rho^{-1/2})(R^{-3/2}\tau_0^{-1/2}), \; G = (\rho^{3/2}\nu g)(R^{-1/2}\tau_0^{-3/2}) \quad (17)$$

Equation (17) shows that increasing R or τ_0 decreases both T and G, and consequently ξ_1 and ξ_2, and thereby increases the growth rate in Eq. (13). These results are different than those of Nachtsheim[4] who chose to fix T and G as R varied which forced the external flow parameter τ_0 to be a function of ν.

Subsonic External Flow

Letting $\chi = 0$, substituting for Λ_i and Λ_r from Eq. (12) into Eq. (10), and setting $c_i = 0$ give the cut-off wavenumbers

$$\alpha_c = \frac{1}{2RTc_f(1 - M^2)^{1/2}} \pm \left[\frac{1}{4R^2T^2c_f^2(1 - M^2)} + \frac{\frac{4}{5}\gamma(1 - \gamma) - G_y}{T}\right]^{1/2} \quad (18)$$

As in the supersonic case, the surface tension and liquid viscosity are stabilizing. The normal body force is stabilizing or destabilizing, depending on whether it is directed toward or outward from the liquid. The axial body force is stabilizing if it opposes the external flow; otherwise, it is destabilizing. In the subsonic case, both cut-off wavenumbers can be calculated because they are O(0.1) or less, and hence within the applicability of the present analysis. Since the first cut-off wavenumber is small, the following discussion is directed toward the second cut-off wavenumber. An increase/decrease in the cut-off wavenumber in the subsonic case can be interpreted as destabilizing/stabilizing because it increases/decreases the range of unstable wavenumbers. The effects of R, τ_0, and M on the stability are shown in Fig. 2 for comparison with the supersonic case. In the subsonic case, increasing R, τ_0, or M increases the cut-off wavenumber, and is hence destabilizing. Thus, increasing R and τ is destabilizing in both the subsonic and supersonic cases, whereas increasing M is stabilizing for the supersonic and destabilizing for the subsonic case.

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